SIGGRAPH ASIA 2021 TOKYO LIVE

Spiral-Spectral Fluid Simulation

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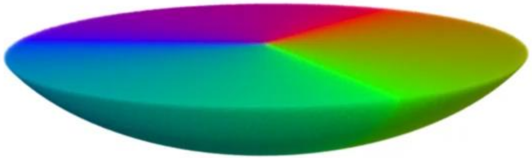
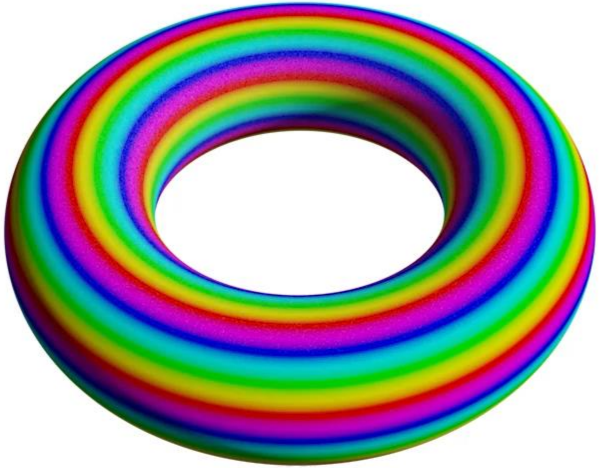
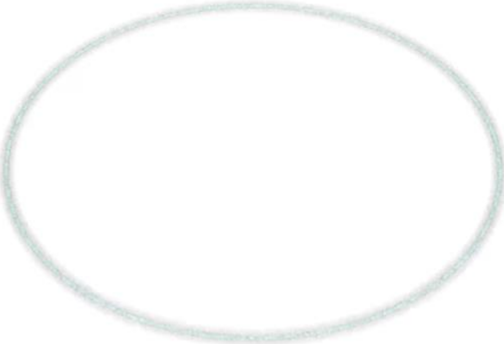
Spheres

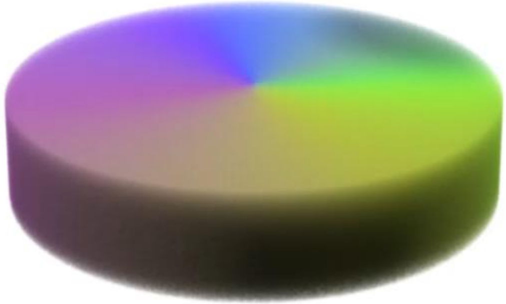


Spheroids

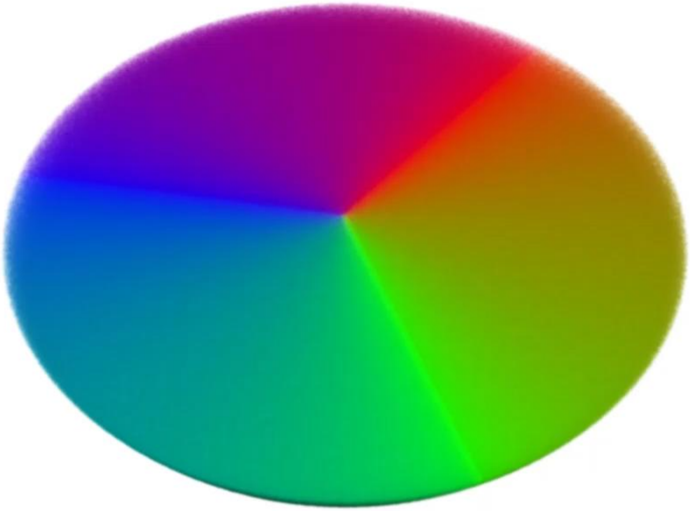




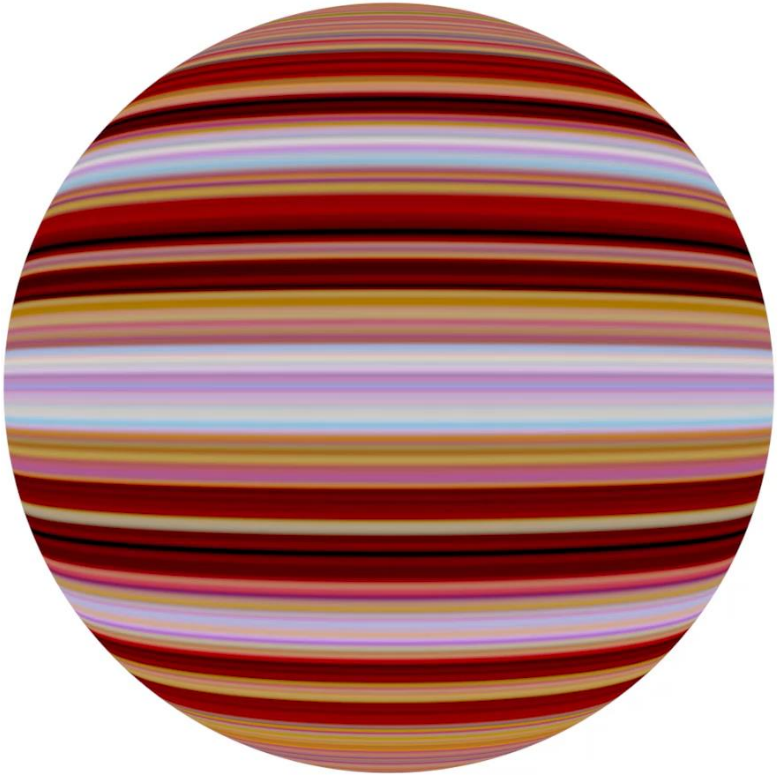
Tori



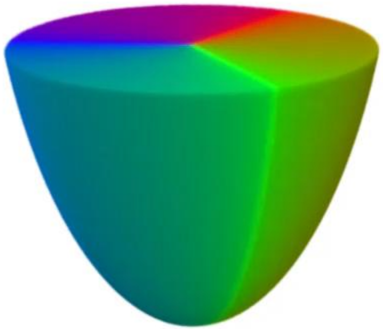
Cylinders

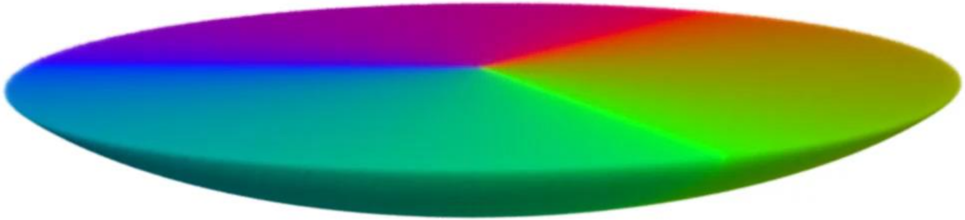


Volumetric sphere

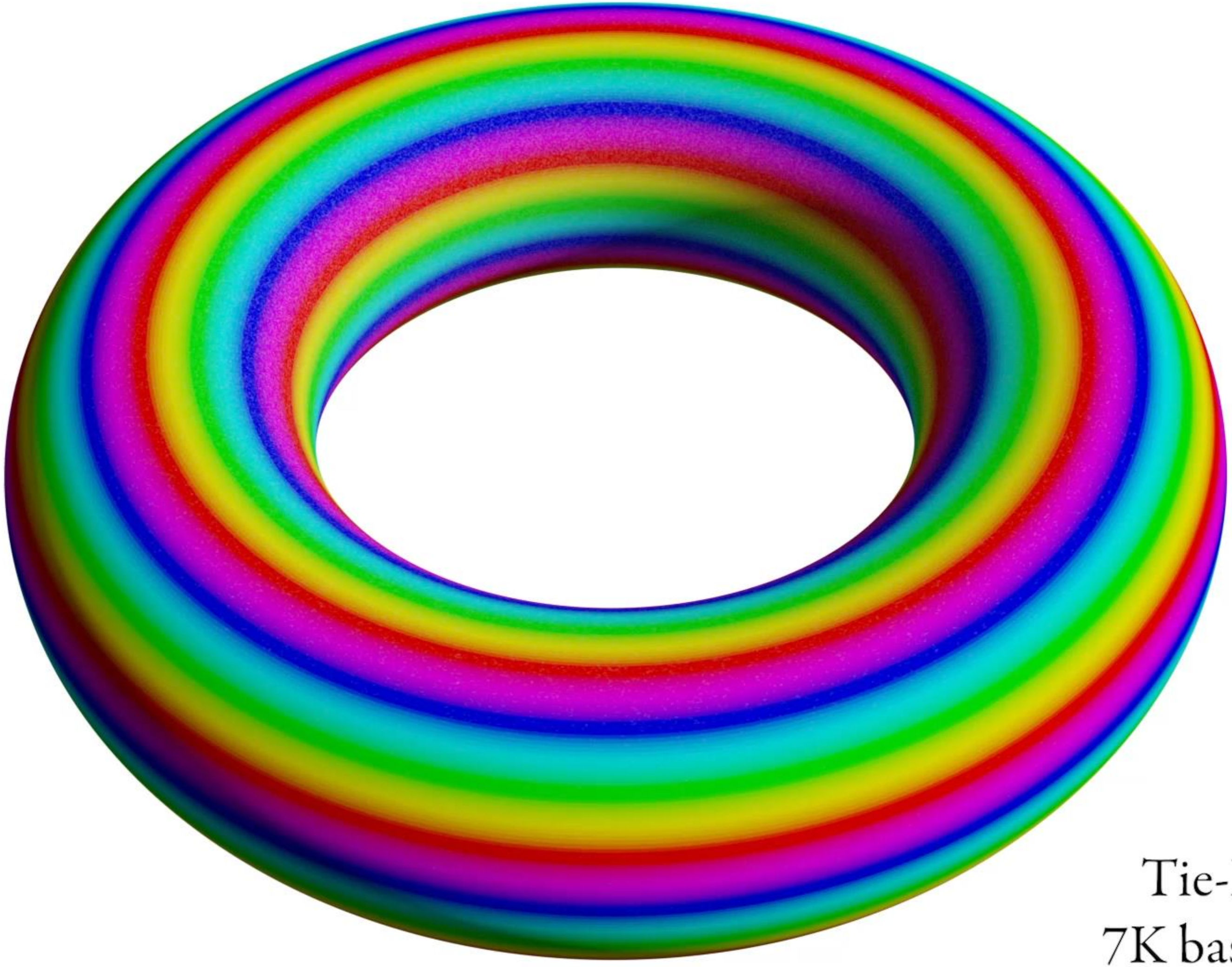


Codimensional sphere





Spheroids (b=0.4), 6K basis functions

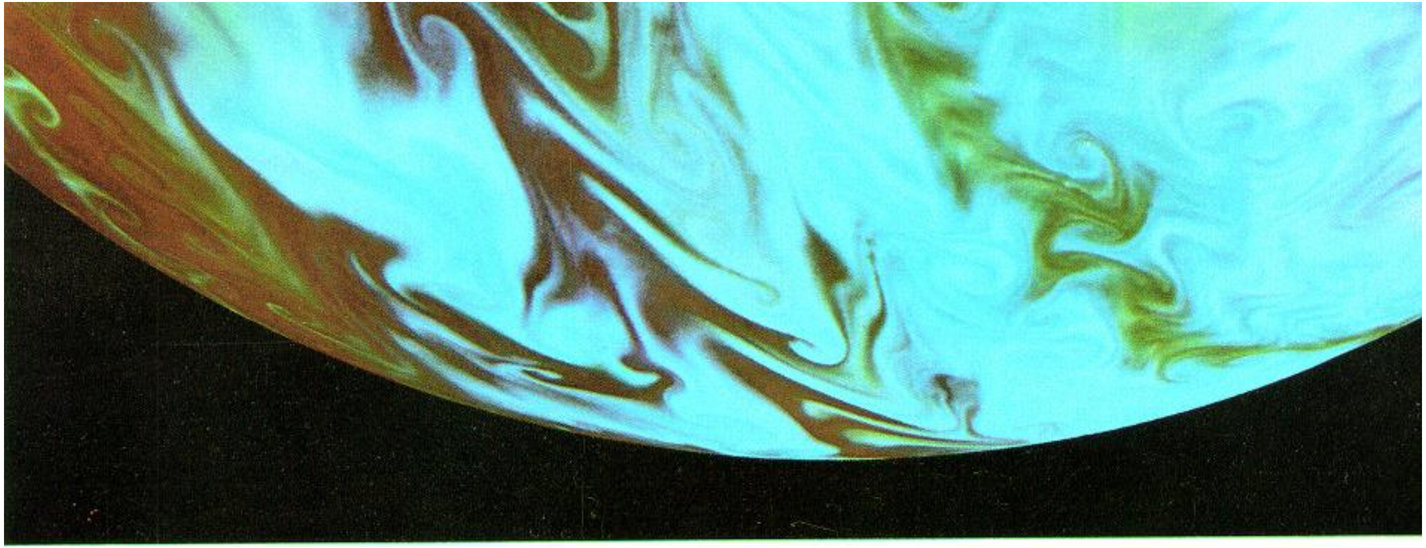


Tie-Dye Torus
7K basis functions

Outline

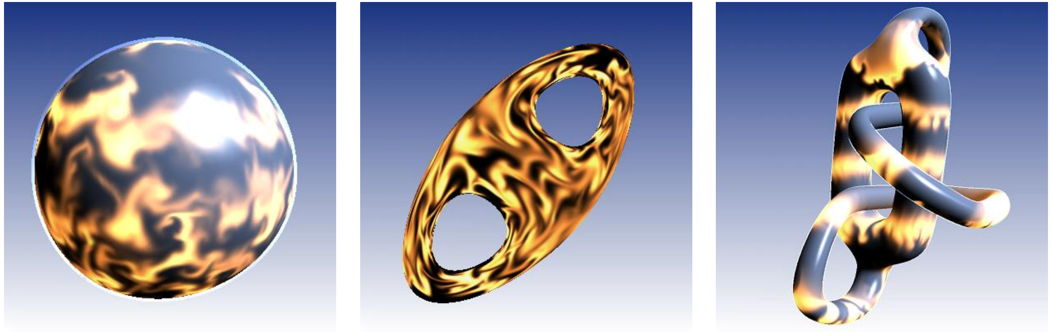
- Previous work
- Scalable Laplacian Eigenfluids
- Our methods
- Results
- Contributions and future work

Simulations in Spherical Coordinates

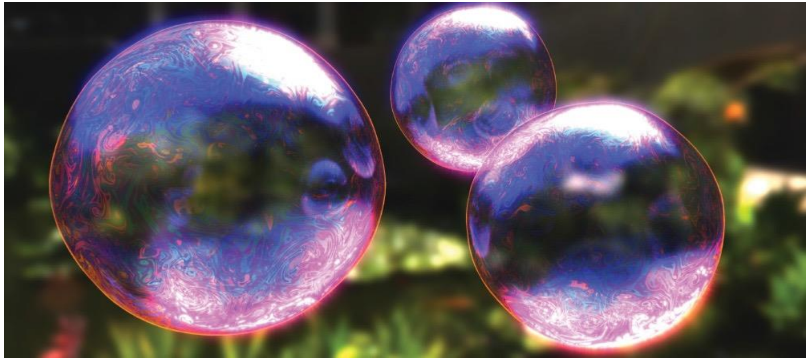


Planetary Flow: [Yaeger et al. 1986]

Simulations in Spherical Coordinates

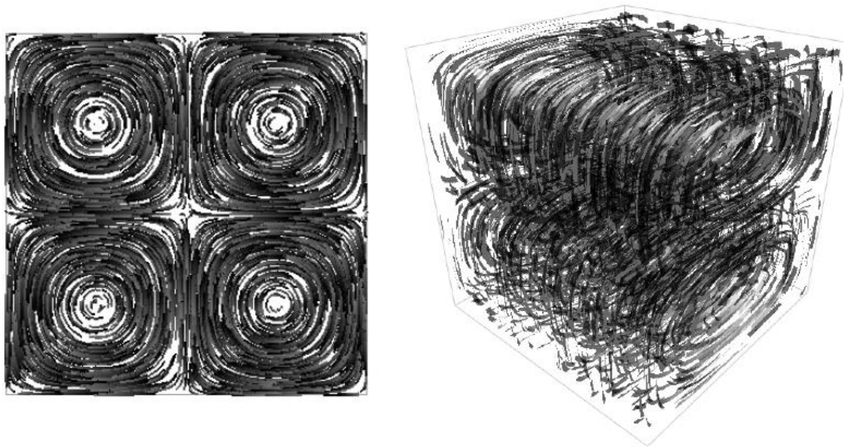


Curvilinear Coordinates: [Stam 2003]

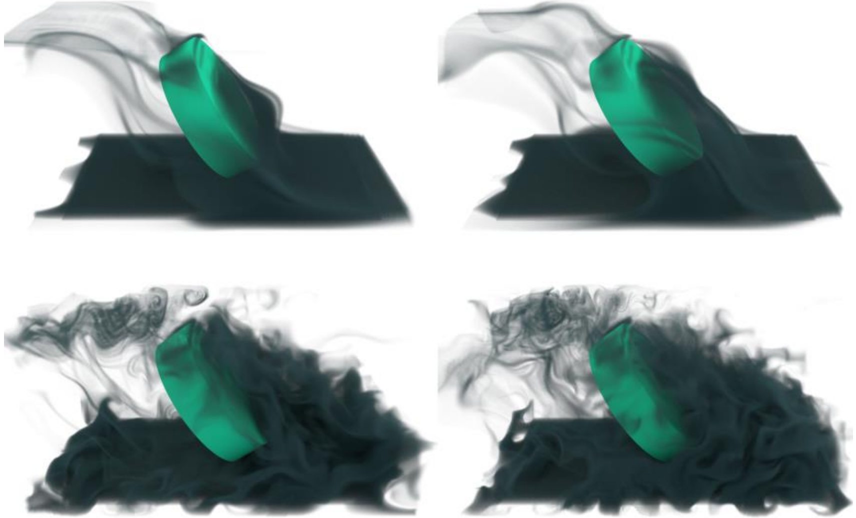


Spherical Coordinates: [Hill et al. 2016]

Scalable Laplacian Eigenfluids



[De Witt et al. 2012]



[Cui et al. 2018]

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Laplacian Eigenfluids

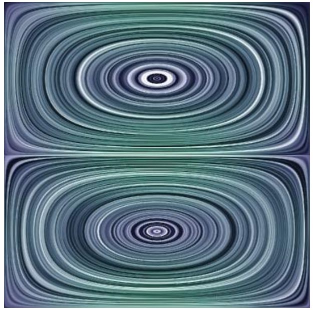
$$\dot{\mathbf{u}} = -\mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u} - \nabla p + \mathbf{f}$$

$$\nabla \cdot \mathbf{u} = 0$$

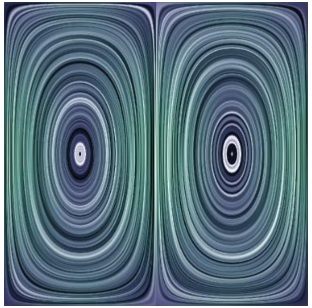
$$\mathbf{u} = \sum_{i=1}^{r} w_i \boldsymbol{\Phi} + w_1 \boldsymbol{\Phi}_1 + w_2 \boldsymbol{\Phi}_2 + w_3 \boldsymbol{\Phi}_3 + \cdots + w_s \boldsymbol{\Phi}_s$$

$$\nabla \cdot \boldsymbol{\Phi}_i = 0$$

$\boldsymbol{\Phi}_1$

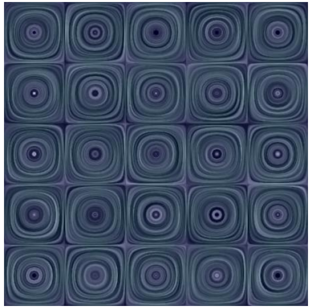


$\boldsymbol{\Phi}_2$



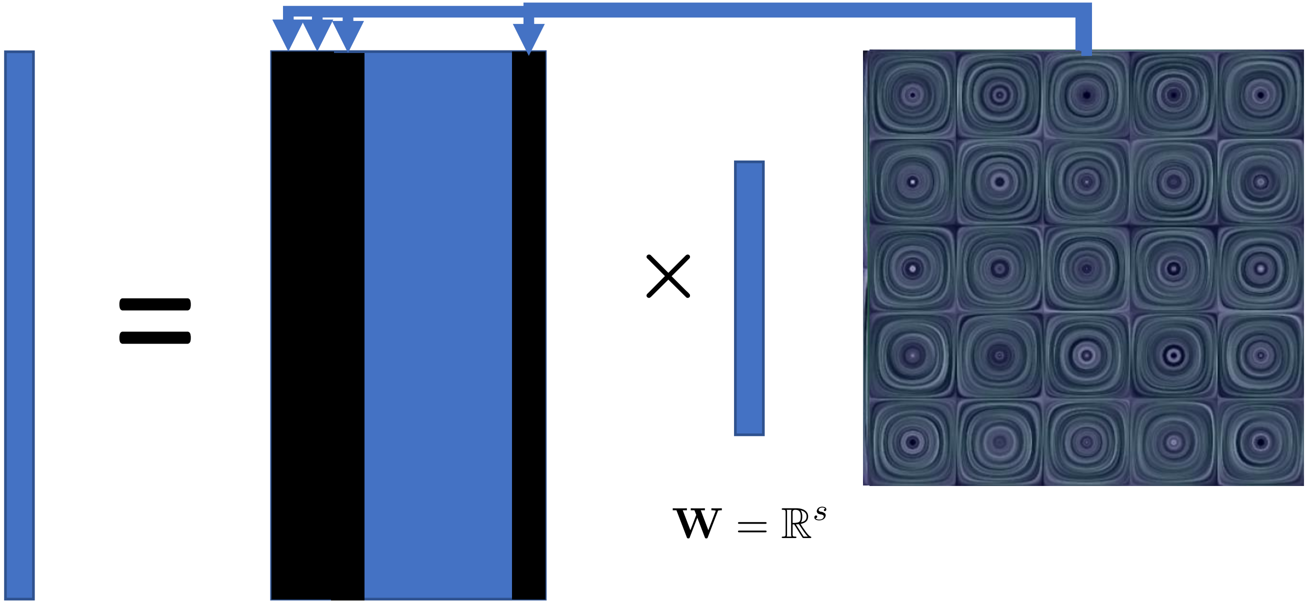
$\boldsymbol{\Phi}_3$

$\cdots$



$\boldsymbol{\Phi}_s$

Reconstruction Bottleneck



$\mathbf{u} \in \mathbb{R}^{N^3}$

$\mathbf{U} \in \mathbb{R}^{N^3 \times s}$

$\mathbf{U} = \{\mathbf{\Phi}_1, \mathbf{\Phi}_2, \ldots, \mathbf{\Phi}_s\}$

$\mathbf{W} = \mathbb{R}^s$

$N = 128$

$s \approx 1000$

50.5 GB

84%

Scalable Laplacian Eigenfluids

- Analytical basis admits fast transformations

$$\Phi_x = -i_2 \sin(i_1 x)\cos(i_2 y)$$
$$\Phi_y = i_1 \cos(i_1 x)\sin(i_2 y)$$

- Memory complexity: $O(sN^3) \longrightarrow O(s)$

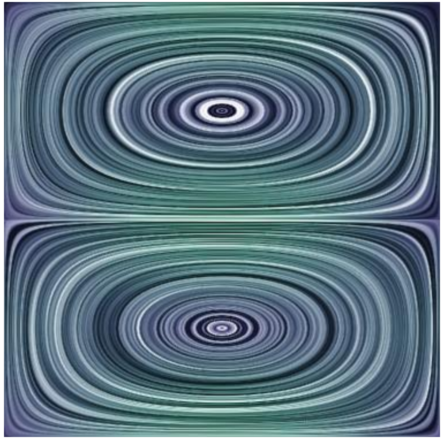
- Time complexity: $O(sN^3) \longrightarrow O(N^3 \log(N))$

- Limited to rectangular domains (Cartesian coordinates)

Cartesian Versus Polar

$$\Phi_x = -i_2 \sin(i_1 x)\cos(i_2 y)$$

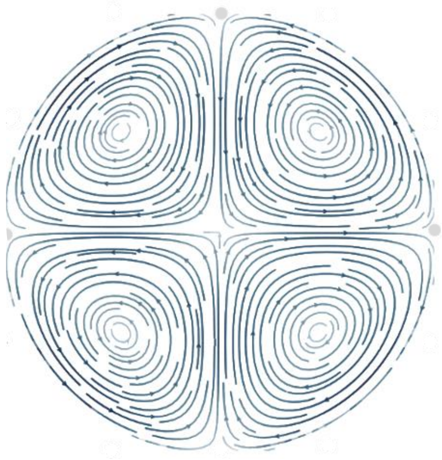
$$\Phi_y = i_1 \cos(i_1 x)\sin(i_2 y)$$



$$\nabla \cdot \mathbf{\Phi} = \frac{\partial \mathbf{\Phi}_x}{\partial x} + \frac{\partial \mathbf{\Phi}_y}{\partial y} = 0$$

$$\Phi_r = ?$$

$$\Phi_\theta = ?$$



$$\nabla \cdot \mathbf{\Phi} = \mathbf{\Phi}_r + r\frac{\partial \mathbf{\Phi}_r}{\partial r} + \frac{\partial \mathbf{\Phi}_\theta}{\partial \theta} = 0$$

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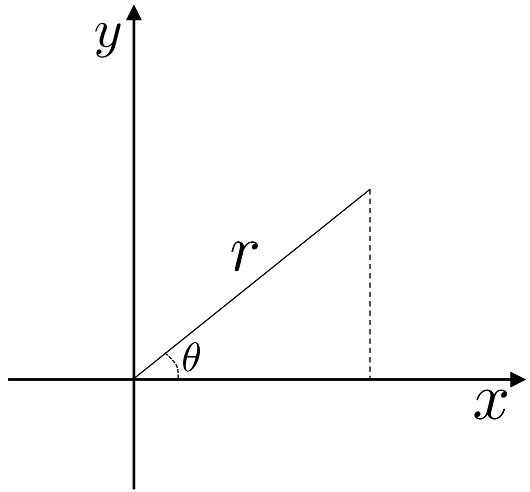
Polar Coordinates



$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

$$\nabla \cdot \mathbf{u} = \frac{1}{r}\left(\mathbf{u}_r + r\frac{\partial \mathbf{u}_r}{\partial r} + \frac{\partial \mathbf{u}_\theta}{\partial \theta}\right)$$

Principles for Good Bases

- Support FFT-based transformations
  - Fast reconstruction & no basis storage
- Separable along each coordinate directions
- Smooth and continuous
- Complete

Eigenfluids in Polar Coordinates

$$\nabla \cdot \mathbf{u} = (\mathbf{u}_r + r\frac{\partial \mathbf{u}_r}{\partial r} + \frac{\partial \mathbf{u}_\theta}{\partial \theta}) = 0$$

$$\mathbf{u}_r = A(r)B(\theta)$$

$$\mathbf{u}_\theta = C(r)D(\theta)$$

$$A(r)\boxed{B(\theta)} + r\frac{dA(r)}{dr}\boxed{B(\theta)} + C(r)\boxed{\frac{dD(\theta)}{d\theta}} = 0$$

$$B(\theta) = \frac{dD(\theta)}{d\theta}$$

Eigenfluids in Polar Coordinates

$$\mathbf{u}_r = A(r)\sin(i_2\theta)$$

$$\mathbf{u}_\theta = C(r)\cos(i_2\theta)$$

$$A(r) + r\frac{dA(r)}{dr} + C(r) = 0$$

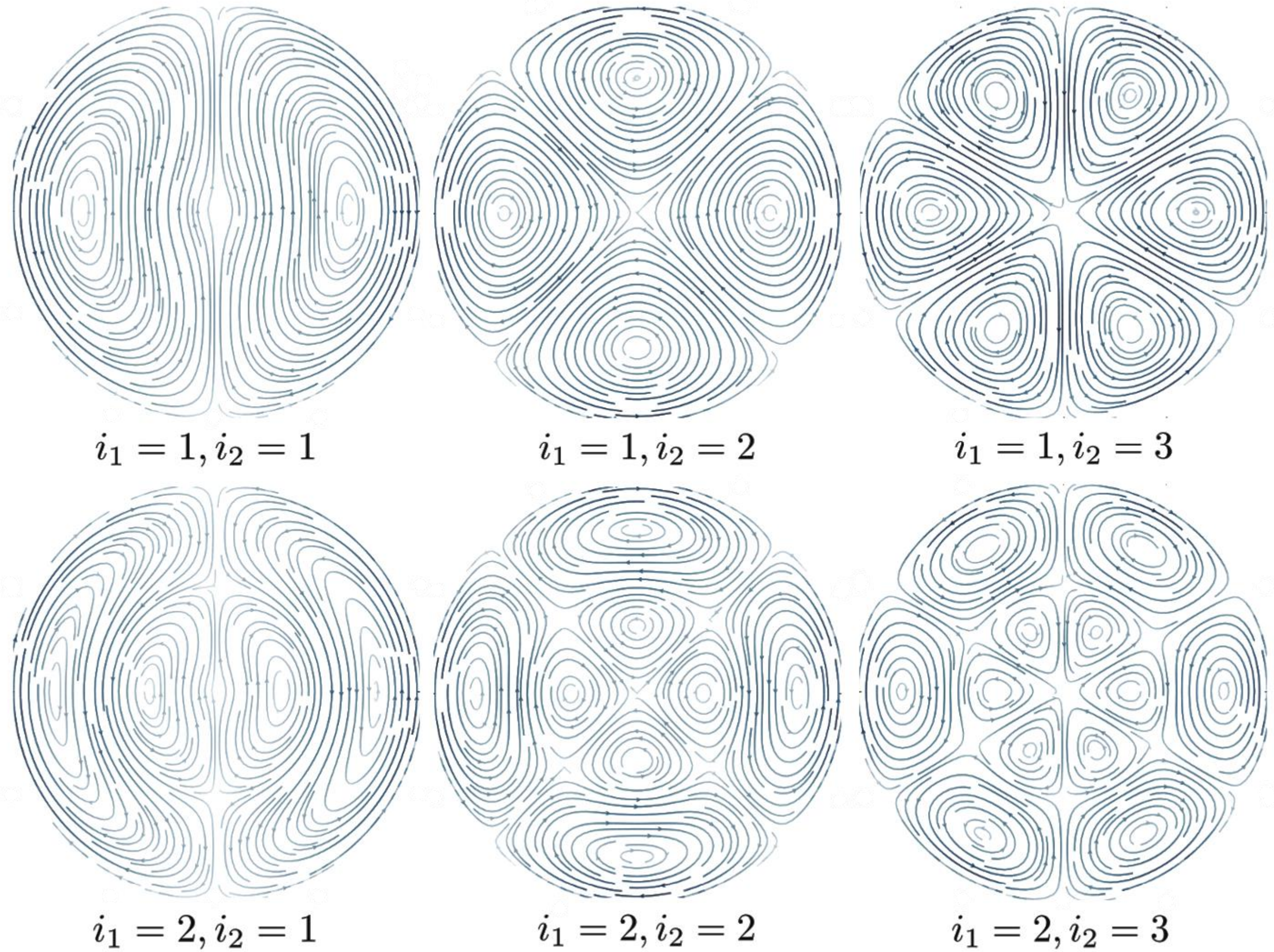
$$A(r) = \sin(i_1\pi r)$$

The First Basis Function

$$\begin{cases} \mathbf{\Phi}_r^0(r,\theta) &= \sin(i_1\pi r)\sin(i_2\theta) \\ \mathbf{\Phi}_\theta^0(r,\theta) &= \frac{1}{i_2}\left(\sin(i_1\pi r) + i_1\pi r\cos(i_1\pi r)\right)\cos(i_2\theta) \end{cases}$$

$$i_1, i_2 \in \mathbb{Z}^+$$

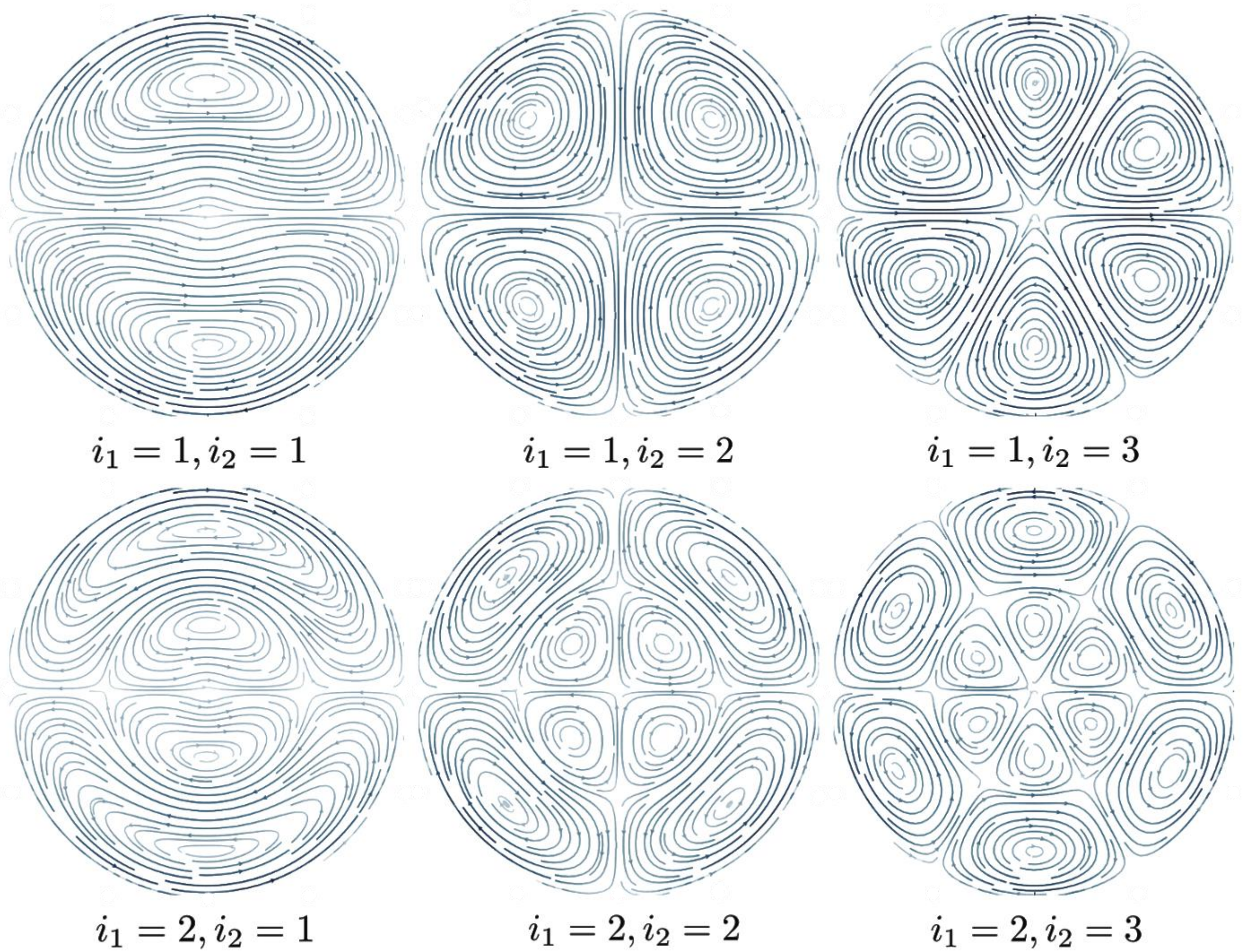
$$r \in [0,1], \theta \in [0, 2\pi]$$


$i_1 = 1, i_2 = 1$
$i_1 = 1, i_2 = 2$
$i_1 = 1, i_2 = 3$
$i_1 = 2, i_2 = 1$
$i_1 = 2, i_2 = 2$
$i_1 = 2, i_2 = 3$

The Second Basis Function

$$\begin{cases} \mathbf{\Phi}_r^0(r,\theta) & = \sin(i_1\pi r)\boxed{\sin(i_2\theta)} \\ \mathbf{\Phi}_\theta^0(r,\theta) & = \frac{1}{i_2}\left(\sin(i_1\pi r) + i_1\pi r\cos(i_1\pi r)\right)\boxed{\cos(i_2\theta)} \end{cases}$$

$$\begin{cases} \mathbf{\Phi}_r^{1}(r,\theta) & = \sin(i_1\pi r)\cos(i_2\theta) \\ \mathbf{\Phi}_\theta^{1}(r,\theta) & = -\frac{1}{i_2}\left(\sin(i_1\pi r) + i_1\pi r\cos(i_1\pi r)\right)\sin(i_2\theta) \end{cases}$$


$i_1 = 1, i_2 = 1$
$i_1 = 1, i_2 = 2$
$i_1 = 1, i_2 = 3$
$i_1 = 2, i_2 = 1$
$i_1 = 2, i_2 = 2$
$i_1 = 2, i_2 = 3$

Reference result

Result with $\mathbf{\Phi}^0$ and $\mathbf{\Phi}^1$

Problems at the Pole

$$\begin{cases} \mathbf{\Phi}_r^0(0,\theta) &= 0 \\ \mathbf{\Phi}_\theta^0(0,\theta) &= 0 \end{cases}$$

$$\begin{cases} \mathbf{\Phi}_r^1(0,\theta) &= 0 \\ \mathbf{\Phi}_\theta^1(0,\theta) &= 0 \end{cases}$$

Enrichment Basis Functions

$$A(r) + r\frac{dA(r)}{dr} + C(r) = 0$$

$$A = \cos(i_1 \pi r),\ \cos(0) = 1$$

$$\begin{cases} \mathbf{\Phi}_r^2 & = \cos(i_1\pi r)\sin(i_2\theta) \\ \mathbf{\Phi}_\theta^2 & = \frac{1}{i_2}\left(\cos(i_1\pi r) - i_1\pi r\sin(i_1\pi r)\right)\cos(i_2\theta) \end{cases}$$

$$\boxed{\begin{cases} \mathbf{\Phi}_r^2(r=0) & = \sin(i_2\theta) \\ \mathbf{\Phi}_\theta^2(r=0) & = \frac{1}{i_2}\cos(i_2\theta) \end{cases}}$$

Boundary Condition at the Pole

$$\frac{\partial \mathbf{u}_r}{\partial \theta} = \mathbf{u}_\theta, \;\; \frac{\partial \mathbf{u}_\theta}{\partial \theta} = -\mathbf{u}_r, \;\; r = 0$$

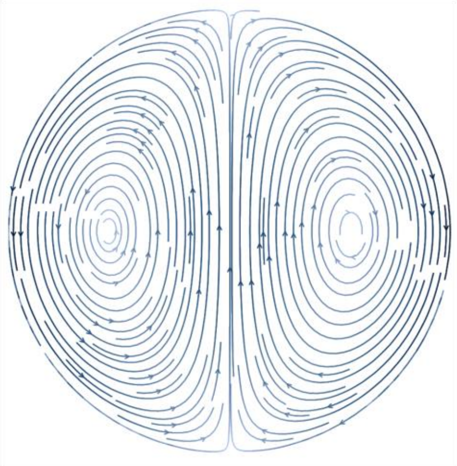
$$\begin{cases} \mathbf{\Phi}_r^2(r=0) & = \sin(\cancel{i_2}\theta) \\ \mathbf{\Phi}_\theta^2(r=0) & = \frac{1}{\cancel{i_2}}\cos(\cancel{i_2}\theta) \end{cases}$$

$$i_2 = 1$$

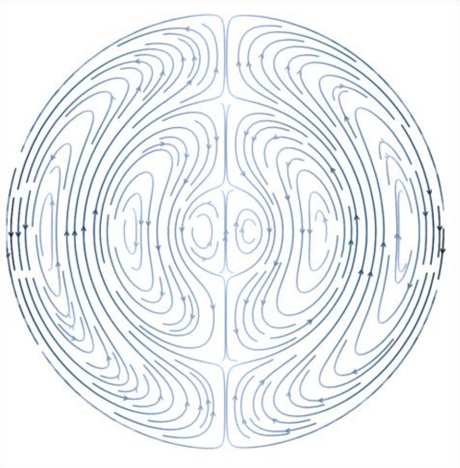
Enrichment Basis Functions

$$\begin{cases} \mathbf{\Phi}_r^2 & = \cos(i_1 \pi r) \sin(\theta) \\ \mathbf{\Phi}_\theta^2 & = (\cos(i_1 \pi r) - i_1 \pi r \sin(i_1 \pi r)) \cos(\theta) \end{cases}$$

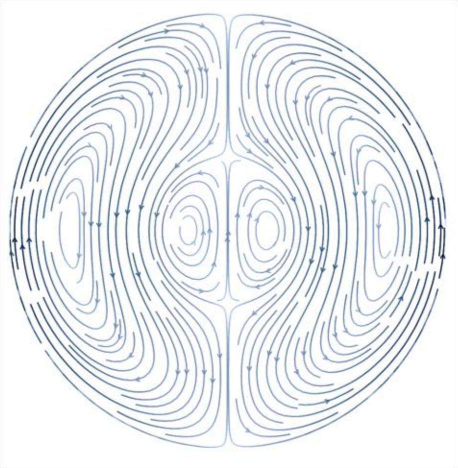
$$\begin{cases} \mathbf{\Phi}_r^3 & = \cos(i_1 \pi r) \cos(\theta) \\ \mathbf{\Phi}_\theta^3 & = (-\cos(i_1 \pi r) + i_1 \pi r \sin(i_1 \pi r)) \sin(\theta) \end{cases}$$



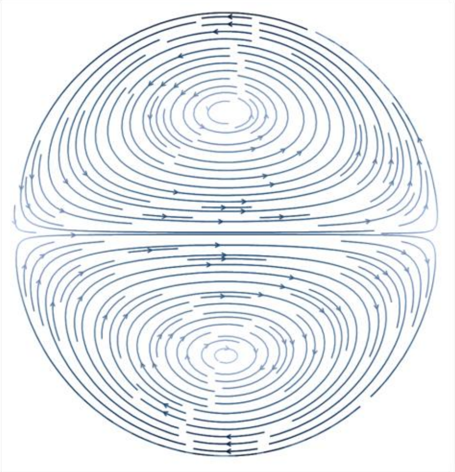
$i_1 = {}^1\!/_2$



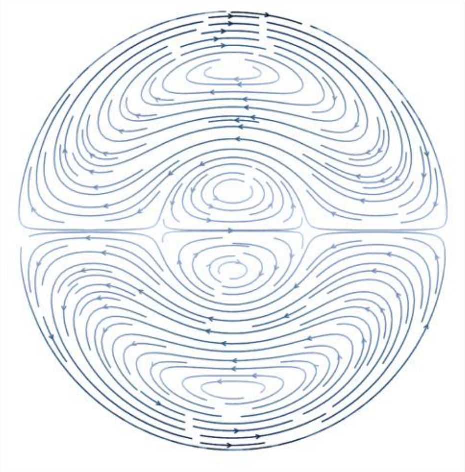
$i_1 = {}^3\!/_2$



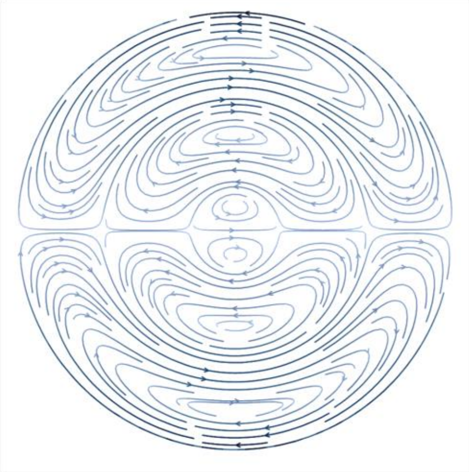
$i_1 = {}^5\!/_2$



$i_1 = {}^1\!/_2$



$i_1 = {}^3\!/_2$



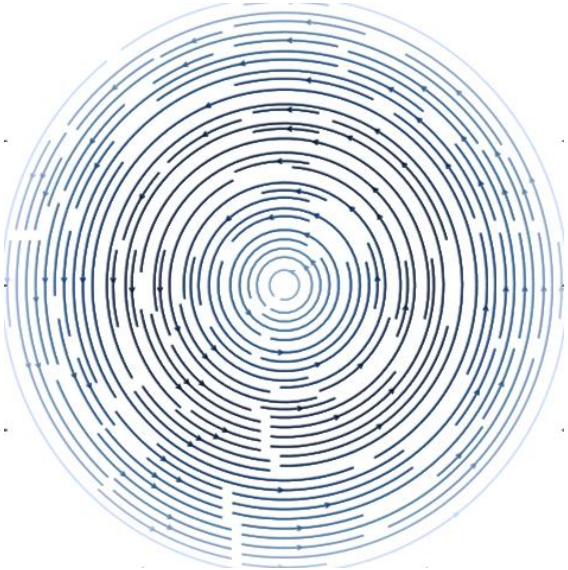
$i_1 = {}^5\!/_2$

Without enrichment functions

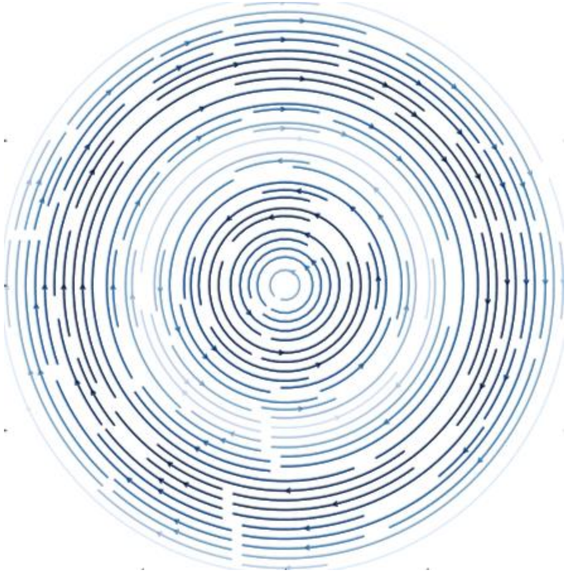
With enrichment functions

Enrichment Basis Functions

$$\begin{cases} \mathbf{\Phi}_r^4 & = 0, \\ \mathbf{\Phi}_\theta^4 & = \sin(i_1 \pi r) \end{cases}$$



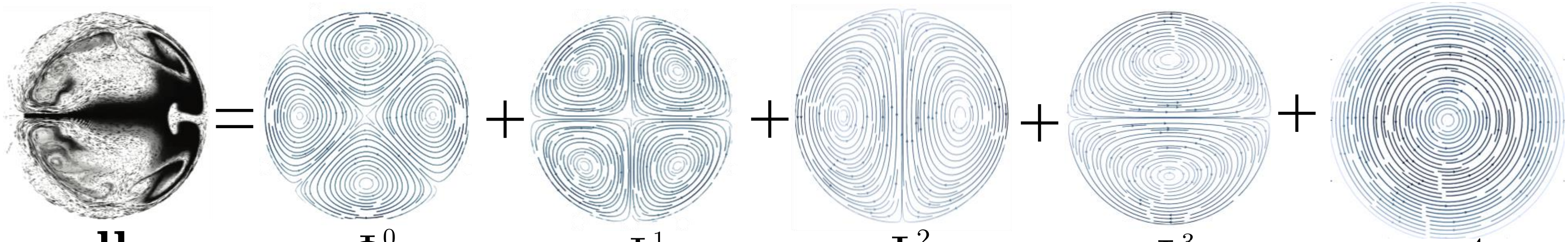
$i_1 = 1$



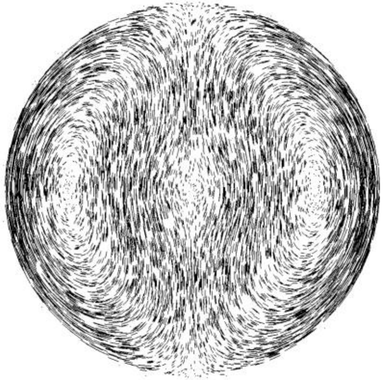
$i_1 = 2$

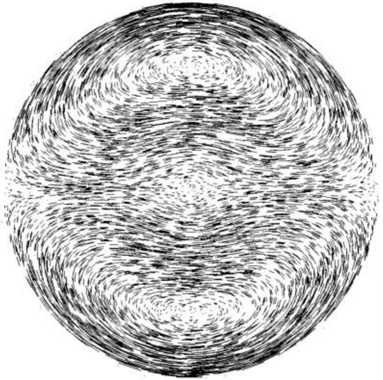
Basis Functions

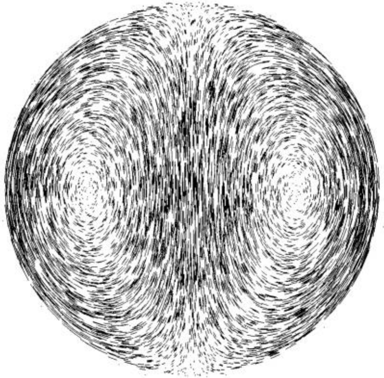
$$\underbrace{\mathbf{\Phi}^0, \mathbf{\Phi}^1}_{\text{Principal functions}} \qquad \underbrace{\mathbf{\Phi}^2, \mathbf{\Phi}^3, \mathbf{\Phi}^4}_{\text{Enrichment functions}}$$

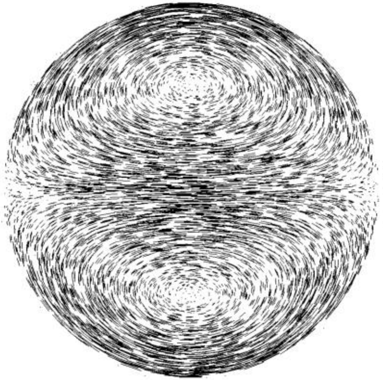


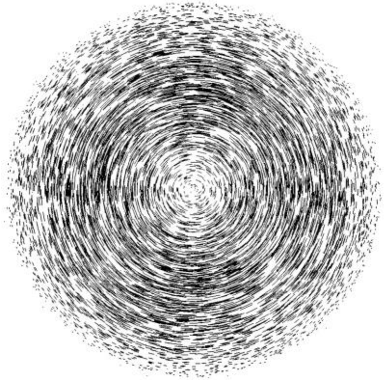
$\mathbf{u} \quad \mathbf{\Phi}^0 \quad \mathbf{\Phi}^1 \quad \mathbf{\Phi}^2 \quad \mathbf{\Phi}^3 \quad \mathbf{\Phi}^4$


$\mathbf{\Phi}^0$


$\mathbf{\Phi}^1$


$\mathbf{\Phi}^2$

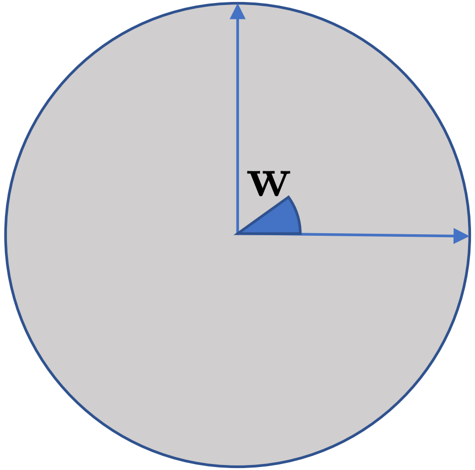

$\mathbf{\Phi}^3$


$\mathbf{\Phi}^4$

Fast Transformations

$$\begin{cases} \mathbf{\Phi}_r^0(r,\theta) & = \sin(i_1\pi r)\sin(i_2\theta) \\ \mathbf{\Phi}_\theta^0(r,\theta) & = \frac{1}{i_2}\left(\sin(i_1\pi r) + i_1\pi r\cos(i_1\pi r)\right)\cos(i_2\theta) \end{cases}$$

$$\mathbf{u}_r = \sum_{i=1}^{r} w_i \sin(i_1\pi r)\cos(i_2\theta)$$



Fast Transformations

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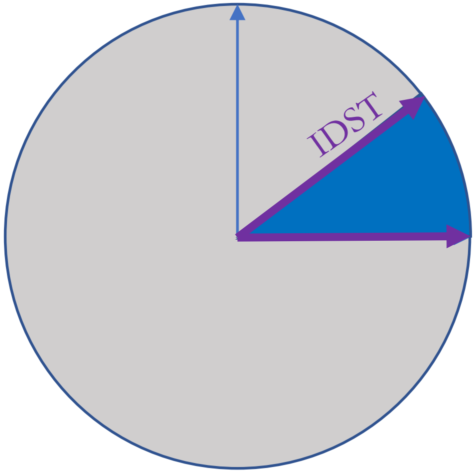
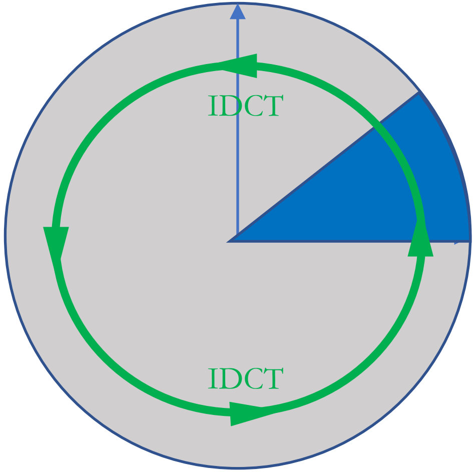
$$\mathbf{u}_r = \sum_{i=1}^{r} w_i \boxed{\sin(i_1\pi r)} \cos(i_2\theta)$$



Fast Transformations

$$\begin{cases} \mathbf{\Phi}_r^0(r,\theta) & = \sin(i_1\pi r)\sin(i_2\theta) \\ \mathbf{\Phi}_\theta^0(r,\theta) & = \frac{1}{i_2}\left(\sin(i_1\pi r) + i_1\pi r\cos(i_1\pi r)\right)\cos(i_2\theta) \end{cases}$$

$$\mathbf{u}_r = \sum_{i=1}^{r} w_i \sin(i_1\pi r)\boxed{\cos(i_2\theta)}$$



Fast Transformations

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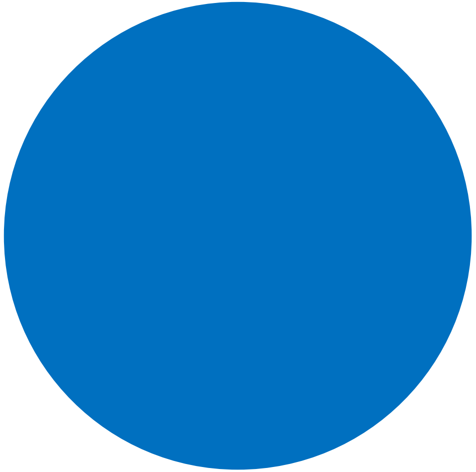
$$\mathbf{u}_r = \sum_{i=1}^{r} w_i \sin(i_1\pi r)\,\boxed{\cos(i_2\theta)}$$

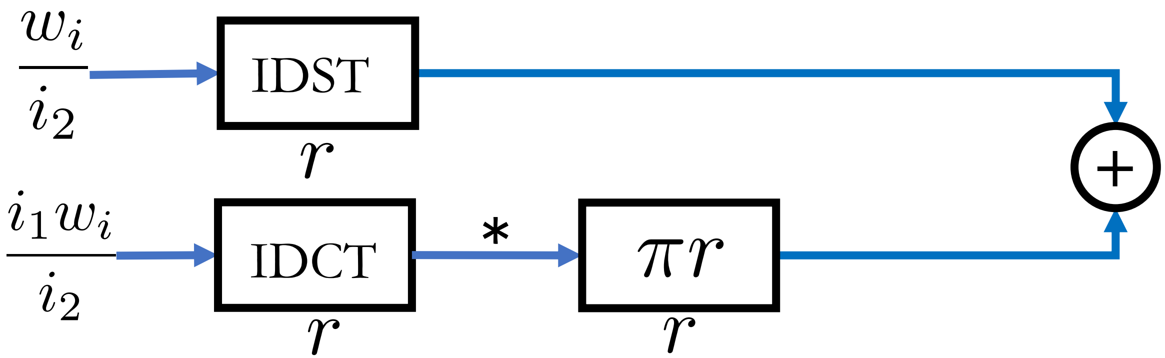


$\mathbf{u}_r$

Fast Transformations

$$\begin{cases} \mathbf{\Phi}_r^0(r,\theta) & = \sin(i_1\pi r)\sin(i_2\theta) \\ \mathbf{\Phi}_\theta^0(r,\theta) & = \frac{1}{i_2}\left(\sin(i_1\pi r) + i_1\pi r\cos(i_1\pi r)\right)\cos(i_2\theta) \end{cases}$$

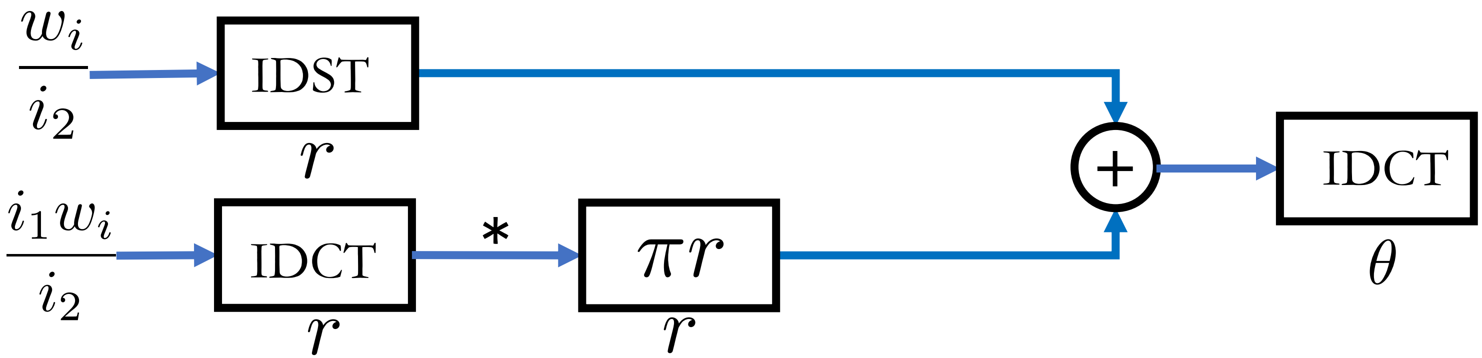
$$\mathbf{u}_\theta = \sum_{i=1}^{r} \boxed{\frac{w_i}{i_2}\left(\sin(i_1\pi r) + i_1\pi r\cos(i_1\pi r)\right)} \cos(i_2\theta)$$



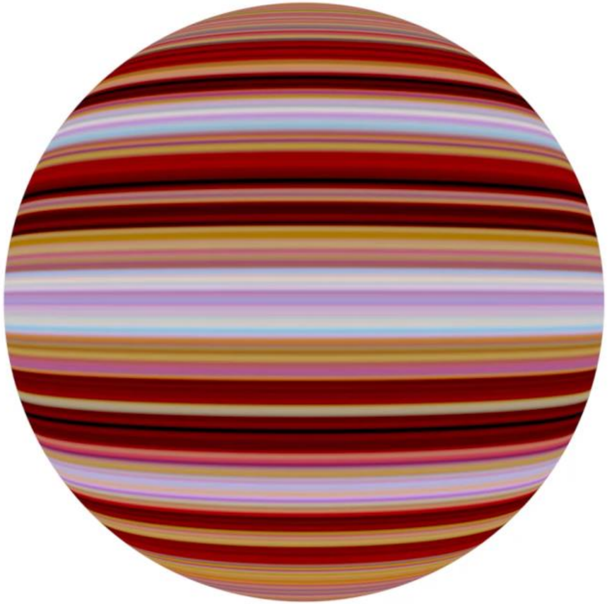
Fast Transformations

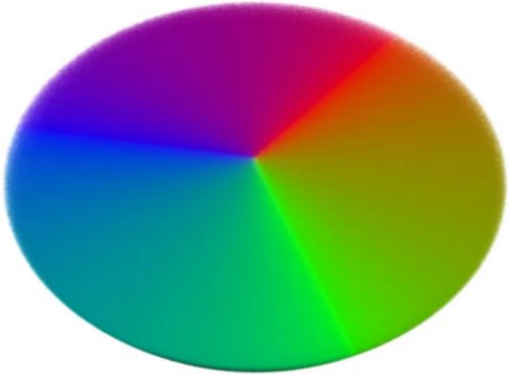
$$\begin{cases} \mathbf{\Phi}_r^0(r,\theta) & = \sin(i_1 \pi r)\sin(i_2\theta) \\ \mathbf{\Phi}_\theta^0(r,\theta) & = \frac{1}{i_2}\left(\sin(i_1\pi r) + i_1 \pi r\cos(i_1 \pi r)\right)\cos(i_2\theta) \end{cases}$$

$$\mathbf{u}_\theta = \sum_{i=1}^{r} \frac{w_i}{i_2}\left(\sin(i_1\pi r) + i_1\pi r\cos(i_1\pi r)\right)\boxed{\cos(i_2\theta)}$$

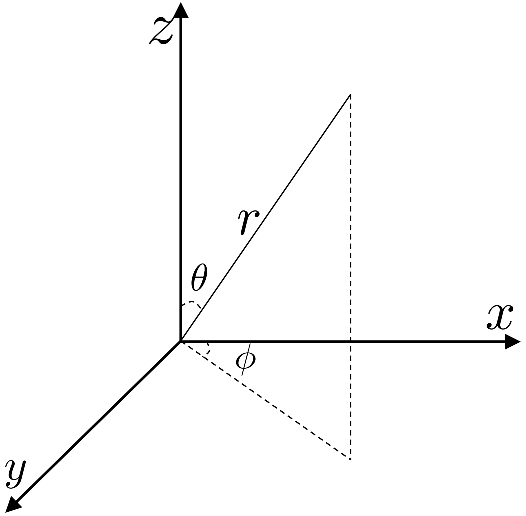


Eigenfluids in Spherical Coordinates





Eigenfluids in Spherical Coordinates



$$x = r\sin(\theta)\cos(\phi)$$
$$y = r\sin(\theta)\sin(\phi)$$
$$z = r\cos(\theta)$$

Eigenfluids in Spherical Coordinates

$$\nabla \cdot \mathbf{u} = \cancel{\frac{1}{r^2}\frac{\partial(r^2\mathbf{u}_r)}{\partial r}} + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}(\mathbf{u}_\theta \sin(\theta)) + \frac{1}{r\sin(\theta)}\frac{\partial \mathbf{u}_\phi}{\partial\phi}$$

Codimensional spherical coordinates

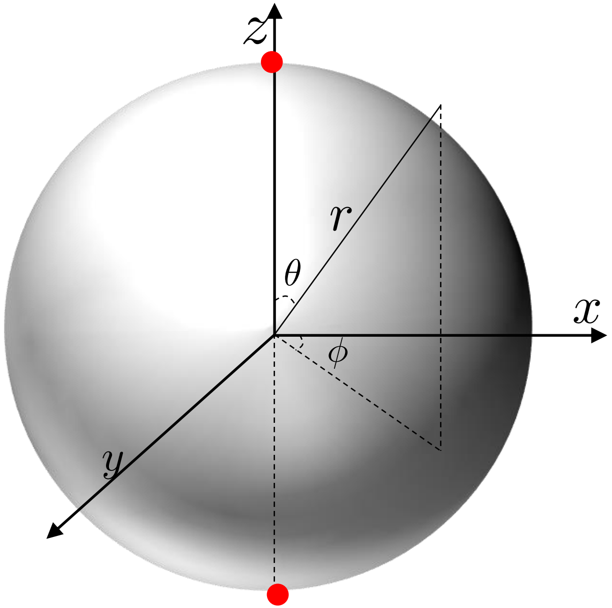
$$\mathbf{u}_r = 0$$

Basis Functions

$$\begin{cases} \mathbf{\Phi}^0_\theta & = \sin(i_1\theta)\cos(i_2\phi),\ \ i_1 > 0,\ \ i_2 > 0, \\ \mathbf{\Phi}^0_\phi & = -\frac{1}{i_2}\left(\cos(\theta)\sin(i_1\theta) + i_1\sin(\theta)\cos(i_1\theta)\right)\sin(i_2\phi) \end{cases}$$

$$\begin{cases} \mathbf{\Phi}^1_\theta & = \sin(i_1\theta)\sin(i_2\phi),\ \ i_1 > 0,\ \ i_2 > 0, \\ \mathbf{\Phi}^1_\phi & = \frac{1}{i_2}\left(\cos(\theta)\sin(i_1\theta) + i_1\sin(\theta)\cos(i_1\theta)\right)\cos(i_2\phi) \end{cases}$$

Boundary Condition at the Pole



Poles:

$$\theta = 0, \ \theta = \pi$$

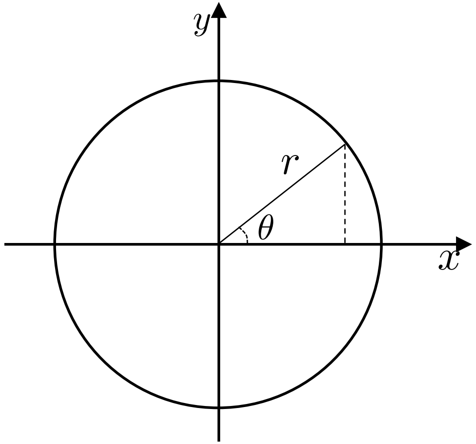
Enrichment Basis Functions

$$\begin{cases} \mathbf{\Phi}_\theta^2 & = \cos(i_1\theta)\cos(\phi) \\ \mathbf{\Phi}_\phi^2 & = (-\cos(\theta)\cos(i_1\theta) + i_1\sin(\theta)\sin(i_1\theta))\sin(\phi) \end{cases}$$

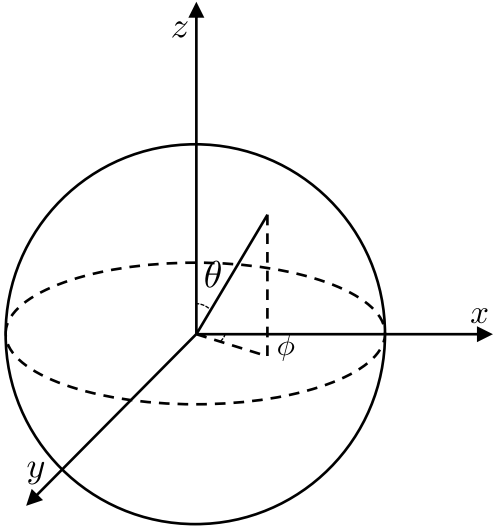
$$\begin{cases} \mathbf{\Phi}_\theta^3 & = \cos(i_1\theta)\sin(\phi) \\ \mathbf{\Phi}_\phi^3 & = (\cos(\theta)\cos(i_1\theta) - i_1\sin(\theta)\sin(i_1\theta))\cos(\phi) \end{cases}$$

Principal basis functions

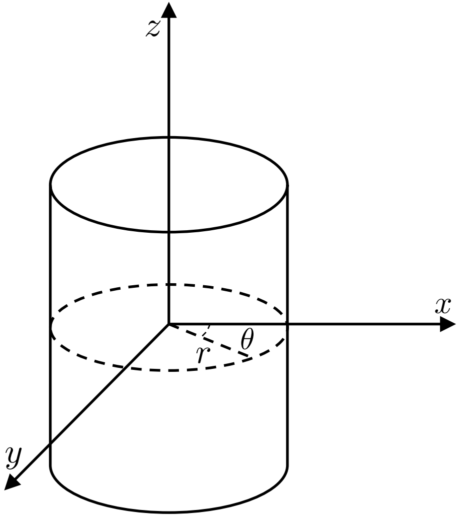
Radial Coordinates



Polar

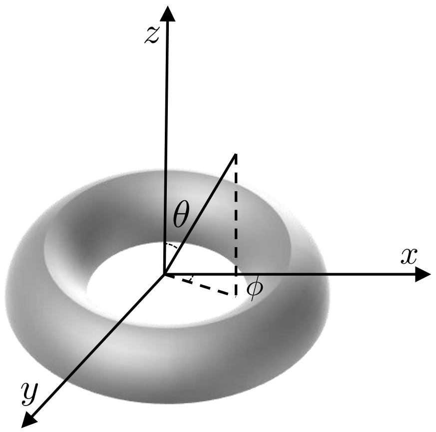


Spherical

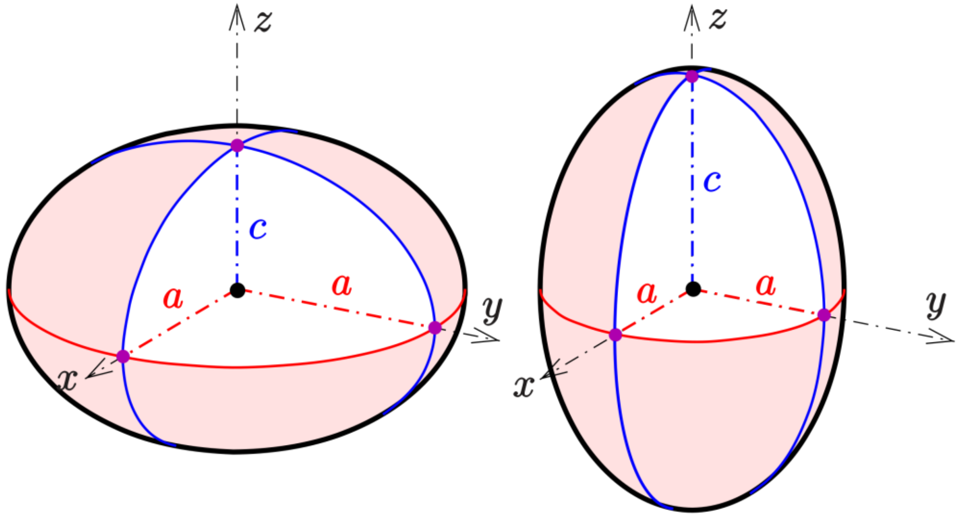


Cylindrical

Radial Coordinates



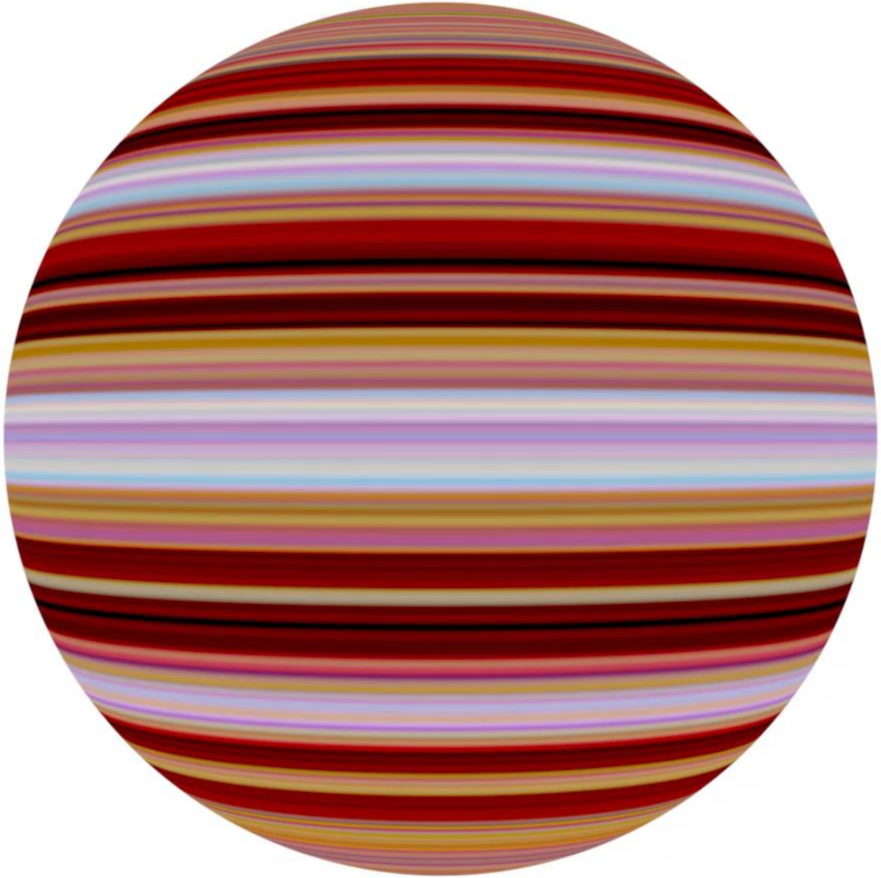
Toroidal



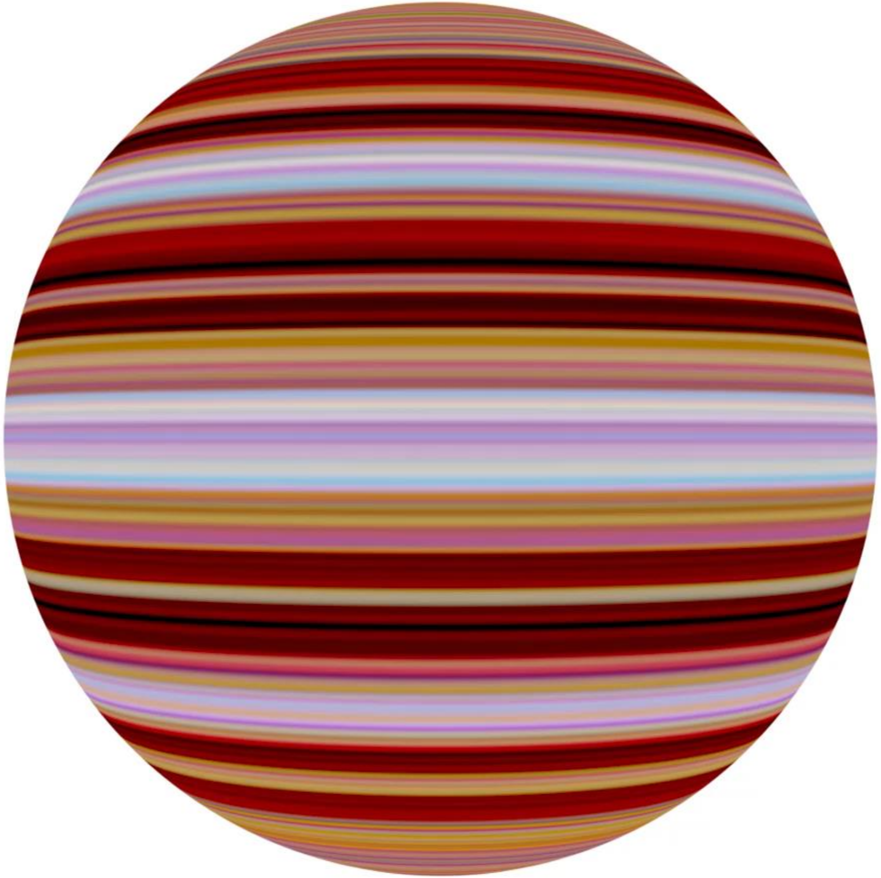
Spheroidal

Outline

- Previous work
- Scalable Laplacian Eigenfluids
- Our methods
- Results
- Contributions and future work



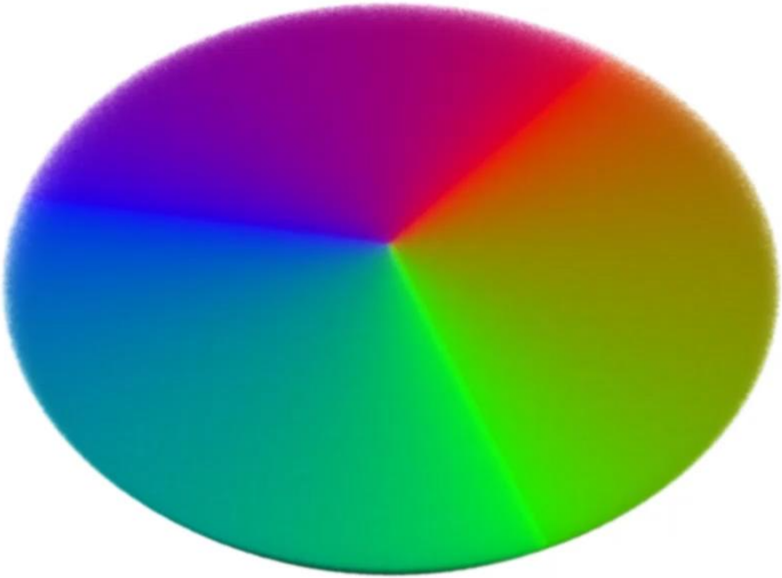
Without Coriolis forces



With Coriolis forces

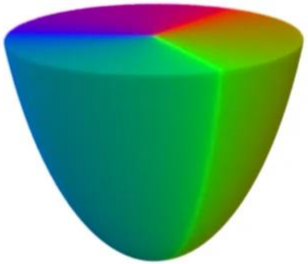
Planetary flow
8K basis functions

3.9 seconds per frame

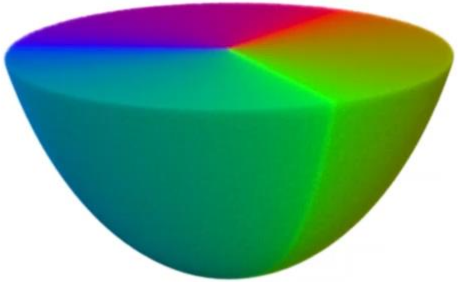


10.8 seconds
per frame

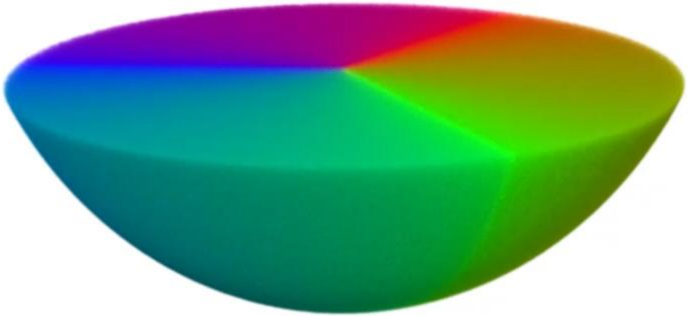
Sphere
6K basis functions



(b=0.4)

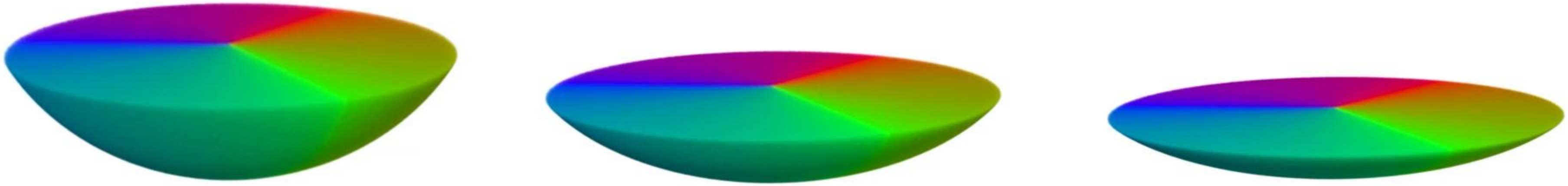


(b=0.6)



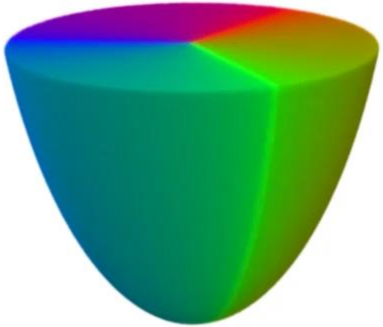
(b=0.9)

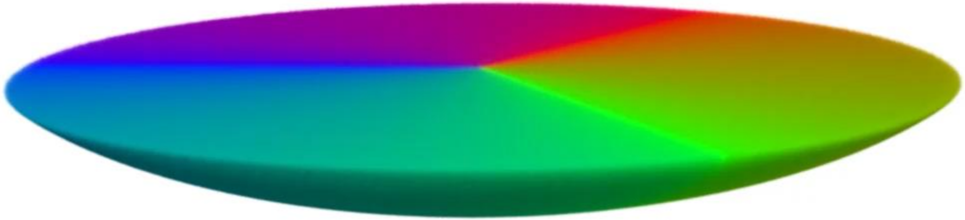
Prolate spheroid, 6K basis functions 10 seconds per frame



(b=0.9) (b=0.6) (b=0.4)

Oblate spheroid, 6K basis functions 12 seconds per frame

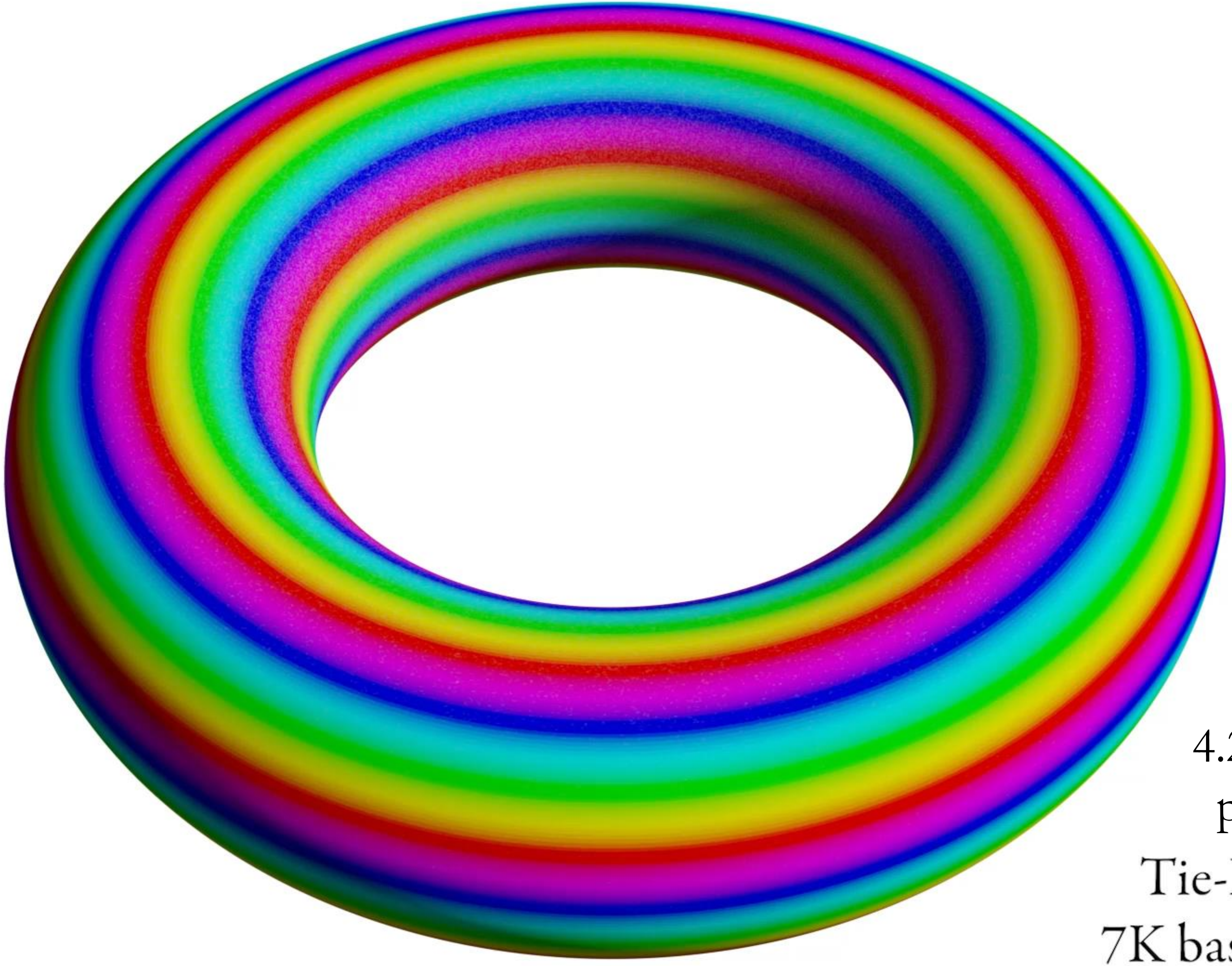




Spheroids (b=0.4), 6K basis functions

6.7 seconds
per frame

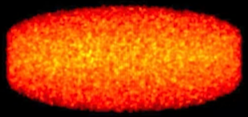
Tokamak Flow
6K basis functions



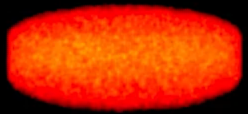
4.2 seconds
per frame
Tie-Dye Torus
7K basis functions

Cylinder tornado

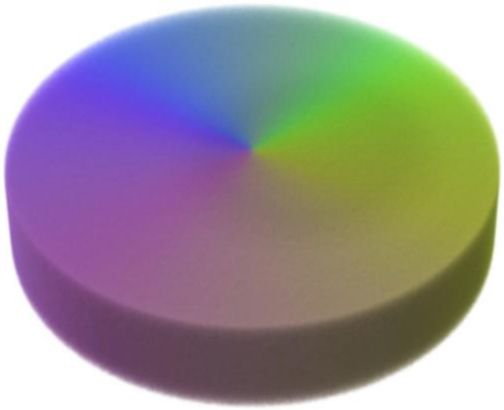
12 seconds per frame



3K basis functions



6K basis functions

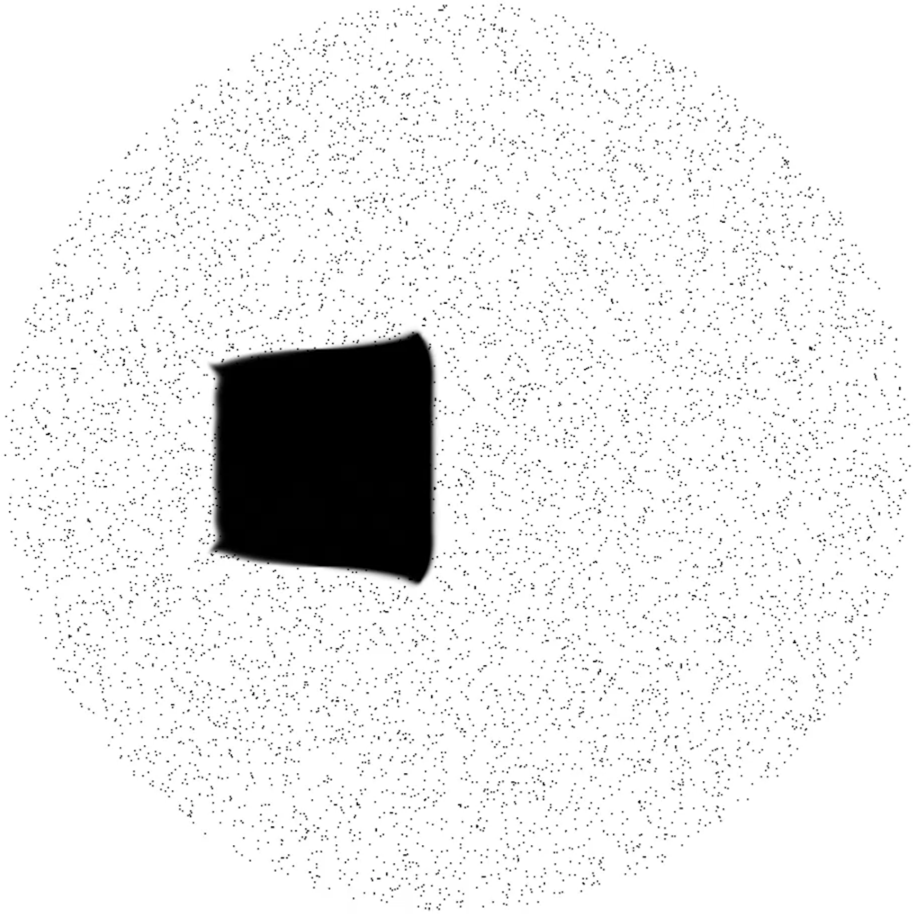


12.9 seconds per frame

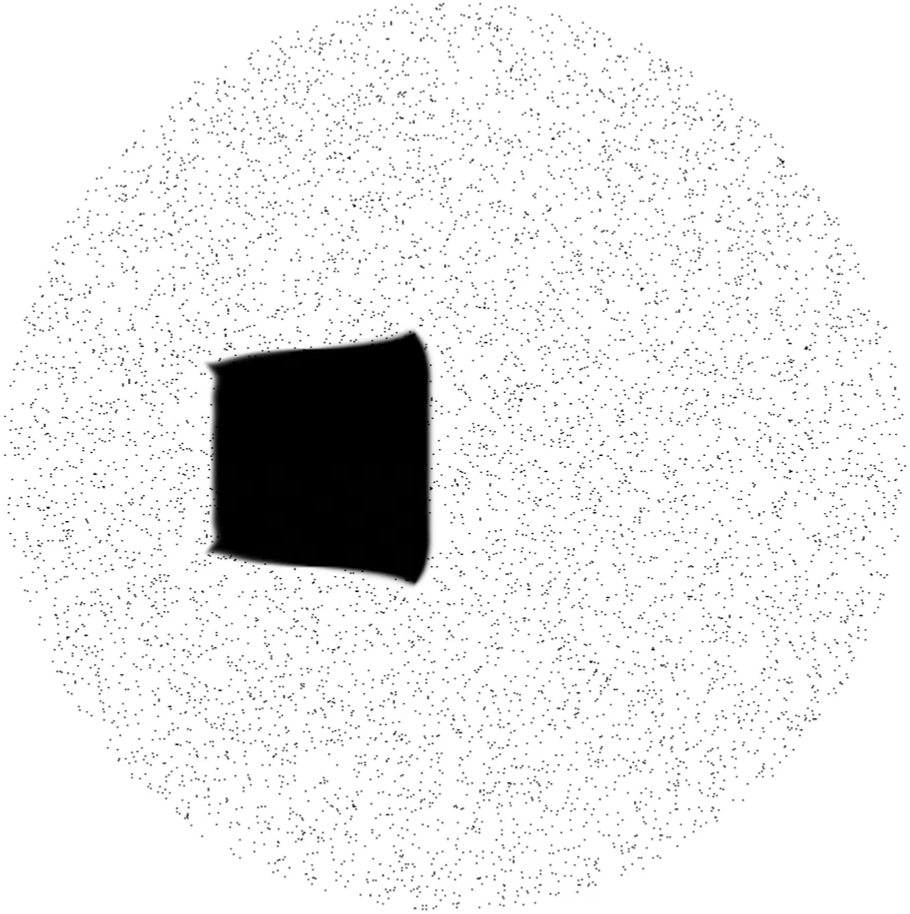
Dropping balls

11K basis functions

visc: 0.003



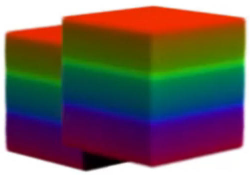
Dirichlet



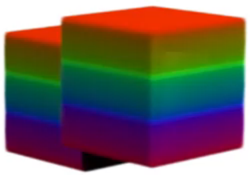
Neumann

Completeness Validation

350 basis functions | 1.5K basis functions | 10K basis functions | Reference



Dedalus



Ours

Outline

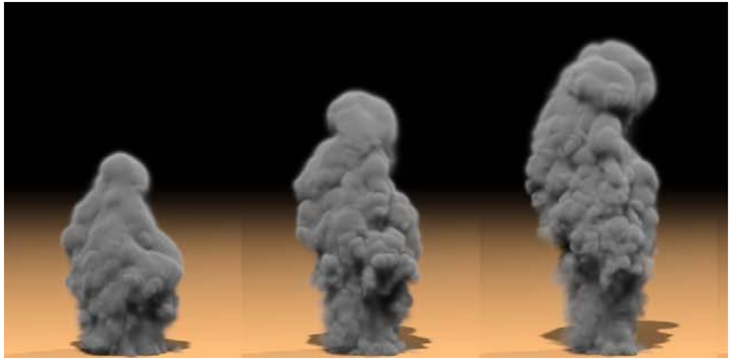
- Previous work
- Scalable Laplacian Eigenfluids
- Our methods
- Results
- Contributions and future work

Contributions

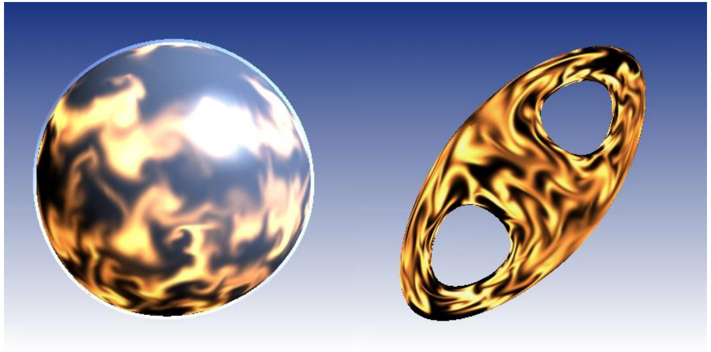
- A method for generating divergence-free basis functions
  - Support FFT
  - Smooth and continuous
  - Analytic
- Efficacy over many geometries
  - Spheres and Spheroids
  - Cylinders
  - Tori

Future Work

- Fluid up-sampling
- Curvilinear coordinates



Vortex Particles: [Selle et al. 2005]



Curvilinear Coordinates: [Stam 2003]

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Thank you

Qiaodong Cui